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# Two-Neutrino Five-Photon Scattering at Low Energies



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**Abstract**

We extend earlier constructions of the effective action for neutrino-photon scattering, using the connection between low-energy neutrino-photon and photon-photon scattering together with the known effective Lagrangian describing low-energy photon scattering in QED. We use this effective action to calculate analytic expressions for the low-energy cross section for the (unpolarised) processes $\nu\overline{\nu} \to 5\,\gamma$, $\nu\gamma \to \nu + 4\,\gamma$ and $\gamma\gamma \to \nu\overline{\nu} + 3\,\gamma$. As a byproduct we derive compact expressions for the $N$-body phase-space integrals for massless particles, including those having non-trivial tensor-structure.

## 1. Introduction

Neutrinos and photons may be, with gravitons, the only particles which are massless, or very nearly so. As such, they are the only degrees of freedom which arise at extremely low energies within the vacuum sector (and possibly within other sectors) of the Standard Model (SM). This makes the study of their low-energy interactions a theoretical labo-

ratory for very-low-energy Standard-Model physics. This study may also have practical applications, despite the extremely weak strength of the interactions, because neutrinos play a unique role within the extreme environments found in astrophysics and cosmology. Precisely because of their weak couplings they are often the mediators of dynamically interesting processes, for instance by being responsible for heat and momentum transfer, especially in the late stages of stellar collapse.

$2 \to 2$ processes — like $\nu\overline{\nu} \to \gamma\gamma$ and $\nu\gamma \to \nu\gamma$ — were first studied long ago and were found to be highly suppressed [1],[2]. The suppression arises because Yang's theorem [3] prohibits the coupling of two photons to a state of angular momentum one, and this ensures that the $O(1/M_W^2)$ contribution to the amplitude for these $2 \to 2$ process must be zero (but see [8]). The dominant contribution therefore arises at $O(1/M_W^4)$, making it smaller than the nominally negligible processes which arise at higher order in $\alpha$, such as $\nu\overline{\nu} \to 3\gamma$ and $\nu\gamma \to \nu + 2\gamma$ [4]. This observation has stimulated more detailed studies of these reactions (both at low energies and at energies above the electron mass [5], [6]), as well as generating searches for practical applications of these $2 \to 3$ processes, such as within stars or in neutrino-photon scattering in the presence of magnetic fields [7].

As pointed out in ref [4], the effective action for these $2 \to 3$ processes is related to the known Euler-Heisenberg effective action for four-photon scattering through the replacement of one of the electromagnetic tensors, $F_{\alpha\beta}$, by a neutrino 'field strength', of the form $N_{\alpha\beta} = \partial_\alpha \left(\overline{\nu}\gamma_\beta\, \gamma_L \nu\right) - (\alpha \leftrightarrow \beta)$. In ref. [9], this connection was examined in some detail: a particular combination of Feynman diagrams was employed to explicitly show the mapping between the calculation of the Euler-Heisenberg action and that governing the interactions of neutrino-antineutrino pairs with three photons.

In this note we have two goals. Our main new result is to extend this treatment to the next least complicated case: that of 2 neutrinos interacting with 5 photons. We do so by computing the relevant terms of the low-energy effective neutrino-photon lagrangian, and use them to calculate analytic expressions for the low-energy neutrino-photon scattering cross sections. In the centre-of-mass (CM) we find these to be of order $\sigma \sim \alpha^3\, G_F^2\, E^{18}/(2\pi)^6 m_e^{16}$, where $E$ denotes the CM scattering energies, as compared with

the $2 \to 2$ result: $\sigma \sim 200\,\alpha G_F^4 E^6/\pi^3$. In principle, the numerical factors are such that the $2 \to 5$ processes can dominate the $2 \to 2$ processes for $E \sim m_e$ (which, of course, lies at the limit of validity of the low-energy approximation), although we know of no practical application of this observation.

Our secondary goal is not so much new as it is explanatory. Preparatory to describing the above results we recast the argument for the suppression of the $2 \to 2$ processes into a more modern effective-lagrangian language. We also rederive the connection between the electromagnetic and neutrino scattering processes within this context. Although these are old results, we hope that their recasting in this way may suggest more applications elsewhere.

Our presentation is as follows. In the next section, §2, we review the low-energy limit of neutrino-photon scattering, rederiving both the connection to the Euler-Heisenberg effective lagrangian and the suppression of $2 \to 2$ processes. This is followed in §3 by the derivation of the low-energy effective action for $2 \to 5$ neutrino-photon interactions. §4 then applies this action to compute the three $2 \to 5$ cross sections: $\nu\overline{\nu} \to 5\gamma$, $\nu\gamma \to \nu + 4\gamma$ and $\gamma\gamma \to \nu\overline{\nu} + 3\gamma$. We conclude in the last section, §5, with comments and final remarks. Finally, an appendix describes an efficient method for computing the relevant $N$-body phase-space integrals which are encountered.

## 2. Low-Energy Neutrino-Photon Scattering Revisited

Because the SM contains no direct (tree-level) couplings between neutrinos and photons, the starting point for calculating their very-low-energy [1] interactions is the SM description of their couplings to other particles. These other particles then generate the effective neutrino-photon interactions once they are integrated out to produce the very-low-energy theory. The effective couplings are nonrenormalizable, in the sense that they are proportional to inverse powers of the masses of the particles which were integrated out to obtain them. Our main interest in what follows is in the dominant interactions at

[1] We use the name 'very-low-energy' to mean energies below the electron mass, in order to distinguish this from other potential notions of 'low energy', such as $E \ll M_W$.

very low energy and so we focus on integrating out the lightest particles. Since the two lightest particles which couple to both neutrinos and photons are electrons and muons, we concentrate our attention on these.

*2.1) The Weak-Scale Effective Theory*

At energies below the $W$-boson mass, the couplings of neutrinos and photons to charged leptons are described by the effective lagrangian obtained from the SM by integrating out the top quark and the electroweak gauge bosons, $W$ and $Z$. The resulting effective interactions which are of most interest in what follows are those which are suppressed by the fewest powers of $M_W$ or $M_Z$. Those involving just neutrinos, photons and charged leptons, obtained by matching to the SM at $\mu = M_W$, are given by:[2]

$$\mathcal{L}_{\rm wk}(\mu = M_W) = e\, A_\mu\, J^\mu_{\rm em} + \frac{G_F}{\sqrt{2}} \sum_{klmn} \Big( i\overline{\nu}_k \gamma_\mu \gamma_L \nu_l \Big)\, L^\mu_{klmn} + O\left(\frac{1}{M_W^4}\right), \tag{1}$$

where $k, l, m, n = e, \mu, \tau$ run over the three lepton flavours, and the charged-lepton currents are given by

$$\begin{aligned} L^\mu_{klmn} &= i\overline{\ell}_m\, \gamma^\mu \left( v_{klmn} + a_{klmn}\gamma_5 \right) \ell_n, \\ J^\mu_{\rm em} &= -\sum_k i\overline{\ell}_k\, \gamma^\mu\, \ell_k. \end{aligned} \tag{2}$$

The effective couplings, $v_{klmn}$ and $a_{klmn}$, as found from tree-level matching are given by:

$$\begin{aligned} & v_{klmn}(\mu = M_W) = \delta_{kn}\,\delta_{lm} + \delta_{kl}\,\delta_{mn} \left( -\frac{1}{2} + 2 s_w^2 \right) \\ \text{and} \qquad & a_{klmn}(\mu = M_W) = \delta_{kn}\,\delta_{lm} - \frac{1}{2}\delta_{kl}\,\delta_{mn}, \end{aligned} \tag{3}$$

at tree level, where $s_w = \sin\theta_w$ is the sine of the weak mixing angle.

Radiative corrections are easily incorporated into this language. Those loops involving high energy degrees of freedom (those involving particles having masses as large as $M_W$ or larger) are included by matching to the SM with higher-loop accuracy. Intermediate-scale

[2] Like all God-fearing people, our conventions are: $\eta_{\mu\nu}$=(−,+,+,+) and $\overline{\psi} = i\psi^\dagger\gamma^0$.

loops (involving particles having masses between $m_e$ and $M_W$) are obtained by running the effective theory down to each new particle threshold, and then matching across this threshold.

One such high-energy loop generates an effective coupling between neutrinos and photons which is proportional to $1/M_W^4$ [4]:

$$\mathcal{L}_{\rm eff}(\mu) = \frac{4\alpha}{\pi M_W^2} \left( \frac{G_F}{\sqrt{2}} \right) \left[ 1 + \frac{4}{3} \ln \left( \frac{M_W^2}{\mu^2} \right) \right] \left( i\overline{\nu}\, \gamma_\alpha \gamma_L \overleftrightarrow{\partial}_\beta\, \nu \right) \; F^{\beta\lambda}\, F^{\alpha}{}_{\lambda}. \tag{4}$$

As we shall see, this particular higher-dimension interaction is *not* generated when lighter particles are integrated out, and so it is the dominant contribution to low-energy $2 \to 2$ photon-neutrino scattering even though it is suppressed by four powers of $M_W$.[3]

Imagine now writing down the effective theory at scale $\mu = m_e$, just before integrating out the electron. The only particles in this low-energy theory are the electron, photon and neutrinos. Conservation of electric charge and lepton numbers require the lowest-dimension neutrino couplings to electrons in this theory to again have the form of eq. (1), although now restricted to electrons and neutrinos. Furthermore, since all of the neutrino interactions in this effective theory must vanish in the limit where the $W$ and $Z$ become infinitely massive, they must be proportional to at least one factor of $G_F$.

It follows that the dominant low-energy interactions in the effective theory at this scale can differ from the electron terms of eq. (1) only through the values taken by the the coefficients $v_{klee}(\mu = m_e)$ and $a_{klee}(\mu = m_e)$. Happily enough, it also happens that $v_{klee}(\mu = m_e)$ cannot differ from its value, eq. (3), at $\mu = M_W$, because the current $\overline{e}\,\gamma^\alpha\, e$ is conserved, and so does not get renormalized.

At low energies the sole contribution of physics between $\mu = M_W$ and $\mu = m_e$ therefore is to the running of the coupling $a_{klee}$ (and of the electric charge, $e$) between these scales, to all orders in all other SM couplings.

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[3] In very-low-energy scattering applications it is the effective couplings renormalized at the electron mass which are required, so $\mu{=}m_e$ is used in eq. (4).

*2.2) Matching at $m_e$*

Next integrate out the electron itself to obtain the effective theory of photons and neutrinos only. At this stage we keep effective interactions having more than the minimal dimension, because these receive their largest coefficients when the lightest possible particle – the electron – is integrated out. We obtain in this way all contributions to low-energy neutrino-photon physics which are $O(1/M_W^2 m_e^p)$, for all $p$.

To this order the new contributions to the effective neutrino-photon interaction lagrangian obtained by matching across the electron mass threshold is therefore given by

$$\mathcal{L}_{\rm elth}(\mu = m_e) = \frac{G_F}{\sqrt{2}} \sum_{kl} \Big( i\overline{\nu}_k \gamma_\mu \gamma_L \nu_l \Big) \, \Big( v_{klee} \, \langle \overline{e} \, \gamma^\mu \, e \rangle + a_{klee} \, \langle \overline{e} \, \gamma^\mu \gamma_5 e \rangle \Big), \tag{5}$$

where $\langle X^\mu \rangle$ represents the expectation of the operator $X^\mu$, obtained by integrating out the electrons, weighted by the QED lagrangian:[4]

$$\langle X^\mu \rangle = \int \mathcal{D}e \mathcal{D}\overline{e} \; X^\mu(e, \overline{e}) \; \exp \left[ i \int d^4x \, \Big( \mathcal{L}_{\rm kin} - ie \, A^\mu \, \overline{e} \, \gamma^\mu \, e \Big) \right]. \tag{6}$$

Eqs. (5) and (6) contain the nub of the main results, because it permits the following two conclusions:

- *Suppression of* $2 \to 2$ *Processes:*

As is easy to show, all operators involving only $\overline{\nu}\gamma^\mu \gamma_L \nu$ – as opposed to $\overline{\nu}\gamma_\alpha \gamma_L \overleftrightarrow{\partial}_\beta \, \nu$ – and two electromagnetic fields vanish on using the equations of motion for the neutrino and photon fields, and so are redundant in the sense that they may be removed by performing a field redefinition. The only possible lowest dimension operator for $2 \to 2$ processes (dimension 6) turns out to be of the form of eq. (4).

[4] A notational aside is in order here, since eq. (5) gives the impression that $\langle X^\mu \rangle$ does not involve an integration over the electromagnetic field as well as the electron field. In reality this expectation denotes the usual matching procedure: the difference between the average calculated with electrons and photons in the theory just above $m_e$, and the average calculated with photons only in the effective theory just below $m_e$. This distinction plays no role in the present discussion.

It remains to show that operators of this form are always suppressed by at least two powers of $G_F$. We have just argued that the right-hand-side of eq. (5) is explicitly proportional to the neutrino current, $\overline{\nu}_k \gamma_\mu \gamma_L \nu_l$, and so cannot contribute to an operator with a derivative embedded within the neutrino bilinear. This is why integrating out the electron does not generate the operator, eq. (4), with a coefficient proportional to $G_F/m_e^2$. The same argument also precludes generating such a term when the other charged leptons are integrated out. Charged-current interactions of neutrinos with quarks, on the other hand, can be linear in $\nu$, and so need not be proportional to $\overline{\nu}_k \gamma_\mu \gamma_L \nu_l$. Nonetheless, conservation of quark flavour only permits these interactions to contribute to neutrino/photon scattering at second order in $G_F$.

- *Connection with Photon-Photon Scattering:*

Since the electromagnetic interactions preserve parity ($\mathcal{P}$) and charge conjugation ($\mathcal{C}$), these symmetries may be used to further organize the contributions to $\mathcal{L}_{\rm elth}$. In particular, these symmetries imply that any term in $\mathcal{L}_{\rm elth}$ involving an odd power of $F_{\mu\nu}$ receives contributions only from the vector current, $\langle \overline{e} \gamma^\mu \, e \rangle$, while those involving even powers of $F_{\mu\nu}$ arise purely from the axial current, $\langle \overline{e} \gamma^\mu \gamma_5 \, e \rangle$.

Furthermore, since the vector current, $\overline{e} \gamma^\mu \, e$, is also the electromagnetic current for the electron effective theory, its expectation may be expressed in terms of the Euler-Heisenberg effective lagrangian, $W_{EH}[A]$, for photon-photon scattering below $m_e$ [4], [9]:

$$\begin{aligned} \langle \overline{e} \gamma^\mu \, e \rangle &= \frac{1}{e} \left( \frac{\delta Z}{\delta A_\mu} \right), \\ \text{where} \qquad Z[A] &= e^{i W_{EH}[A]} = \int \mathcal{D}e \mathcal{D}\overline{e} \; \exp \left[ i \int d^4 x \, \Big( \mathcal{L}_{\rm kin} - ie \, A^\mu \, \overline{e} \, \gamma^\mu \, e \Big) \right]. \end{aligned} \tag{7}$$

For instance, since the quartic contribution to the Euler-Heisenberg interaction is given by: [10]

$$\mathcal{L}_{EH}^{(4)} = \frac{\alpha^2}{180 \, m_e^4} \, \left[ 5 (F_{\mu\nu} F^{\mu\nu})^2 - 14 F_{\mu\nu} F^{\nu\lambda} F_{\lambda\rho} F^{\rho\mu} \right], \tag{8}$$

it follows that the dominant contribution to $\mathcal{L}_{\rm elth}$ involving two neutrinos and three elec-

tromagnetic fields must be [4], [9]:

$$\mathcal{L}^{(3)}_{\rm elth} = \frac{e\,(\frac{1}{2}+2s_w^2)\,\alpha}{90\pi\, m_e^4}\,\left(\frac{G_F}{\sqrt{2}}\right)\Big[5\,(N_{\mu\nu}\,F^{\mu\nu})(F_{\lambda\rho}\,F^{\lambda\rho}) - 14\,(N_{\mu\nu}\,F^{\nu\lambda}\,F_{\lambda\rho}\,F^{\rho\mu})\Big], \qquad (9)$$

with $N_{\alpha\beta} = \partial_\alpha\left(\overline{\nu}\gamma_\beta\gamma_L\nu\right) - (\alpha\leftrightarrow\beta)$.

This method clearly works in general: to obtain any term involving an odd power of $F_{\mu\nu}$ in $\mathcal{L}_{\rm elth}$, replace one power of $e$ by $G_F\left(\frac{1}{2}+2s_w^2\right)/\sqrt{2}$, and sum all possible ways of replacing one electromagnetic field strength by $N_{\mu\nu}$.

*2.3) The $2\to 5$ Effective Lagrangian*

The next simplest neutrino-photon interaction which is related in this way to the Euler-Heisenberg action describes $2\to 5$ processes, like $\overline{\nu}\nu\to 5\gamma$. It is related to the sixth order term of the EH action, which is given by [11]:

$$\mathcal{L}^{(6)}_{EM} = \frac{\pi\alpha^3}{315\; m_e^8}\left[9(F_{\alpha\beta}F^{\alpha\beta})^3 - 26F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}\;(F_{\alpha\beta}F^{\alpha\beta})\right]. \qquad (10)$$

Now, given our previous arguments we can read off the effective two-neutrino/five-photon operators directly. We find:

$$\begin{aligned}\mathcal{L}^{\nu-5\gamma}_{\rm eff} = \;& \frac{\pi}{315}\,\frac{\alpha^{5/2}}{\sqrt{4\pi}}\,\frac{G_F}{\sqrt{2}}\,\frac{\left(\frac{1}{2}+2s_w^2\right)}{m_e^8}\Big[6\cdot 9(F_{\alpha\beta}F^{\alpha\beta})^2F_{\mu\nu}N^{\mu\nu} \\ & - 4\cdot 26F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}N^{\rho\mu}\;(F_{\alpha\beta}F^{\alpha\beta}) - 2\cdot 26F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}\;(F_{\alpha\beta}N^{\alpha\beta})\Big],\end{aligned} \qquad (11)$$

A similar method will also give the even powers of $F_{\mu\nu}$ in $\mathcal{L}_{\rm elth}$ given the expression for the axial-vector/vector current correlation in QED [5] .

[5] After completing this paper it was brought to our attention that, in ref. [13], the expression for $\langle\overline{e}\gamma^\mu\gamma_5 e\rangle$ has been worked out, up to fourth order in the fields. We thank H. Gies for pointing this reference out to us.

## 3. Cross sections for $\overline{\nu}\,\nu \to 5\,\gamma$ and crossed processes

We next apply the lagrangian, eq. (11), to compute the $2 \to 5$ processes $\overline{\nu}\nu \to 5\gamma$, $\nu\gamma \to \nu + 4\gamma$ and $\gamma\gamma \to \overline{\nu}\nu + 3\gamma$. This is a straightforward, if tedious, exercise within the effective theory, requiring only the Born approximation using interaction (11). (This should be contrasted with the difficulty of extracting the low-energy limit of the scattering amplitude, computed directly from the higher-loop graphs involving the weak interaction, eq. (1), and QED. As is usually the case with effective lagrangians, the payoff in simplicity is much larger for nonleading contributions.) In performing this calculation we employed the symbolic manipulation program FORM [14], which reduced the squared amplitude to its final form in under ten minutes on a desktop PC. We briefly sketch the method of computation below.

We require, then, the matrix elements of the effective interaction, eq. (11), which we write as follows:

$$\mathcal{L}_{\text{eff}}^{\nu-5\gamma} = g N^{\mu\nu} T_{\mu\nu}^{\alpha_1\beta_1,\cdots,\alpha_5\beta_5} \partial_{\alpha_1} A_{\beta_1} \cdots \partial_{\alpha_5} A_{\beta_5}$$

where $g$ is the factor premultiplying the square bracket in eq. (11) and $T_{\mu\nu}^{\alpha_1\beta_1,\cdots,\alpha_5\beta_5}$ represents the polynomial of momenta in the effective interaction. In terms of these quantities the matrix element relevant to $\overline{\nu}\nu \to 5\gamma$, for instance, becomes:

$$\langle \gamma_1 \cdots \gamma_5 | \mathcal{L}_{\text{eff}}^{\nu-5\gamma} | \overline{\nu}\nu \rangle = g \widetilde{N}^{\mu\nu} \widetilde{T}_{\mu\nu}^{\alpha_1\beta_1,\cdots,\alpha_5\beta_5} \prod_{i=1}^{5} \left( \frac{k_{(i)\ \alpha_i} \epsilon_{\beta_i}(k_i; \lambda_i)}{\sqrt{(2\pi)^3 k_i^0}} \right), \tag{12}$$

where $\widetilde{N}^{\mu\nu} = \langle 0 | N^{\mu\nu} | \overline{\nu}\nu \rangle$ and

$$\widetilde{T}_{\mu\nu}^{\alpha_1\beta_1,\cdots,\alpha_5\beta_5} = \sum_{\pi \in S_5(1,\cdots,5)} T_{\mu\nu}^{\alpha_{\pi_1}\beta_{\pi_1},\cdots,\alpha_{\pi_5}\beta_{\pi_5}}, \tag{13}$$

is the permutation-summed tensor contracting the fields together. The $\epsilon^\mu$'s are, as usual, the photon polarisation vectors.

After squaring and doing the spin sums, the following phase-space integral is required

to obtain the total cross section:

$$\mathcal{I}_m^{\alpha_1\ldots\alpha_m;\gamma_1\cdots\gamma_m}(w) = \int \frac{d^3k_1}{2k_1^0} \cdots \frac{d^3k_m}{2k_m^0}\, k_1^{\alpha_1} k_1^{\gamma_1} \cdots k_m^{\alpha_m} k_m^{\gamma_m}\ \delta^4(\sum_{i=1}^{m} k_i - w). \tag{14}$$

A general technique for evaluating integrals of this form is given in the Appendix.

- $\overline{\nu}\nu \to 5\gamma$*:* Using the integrals of the Appendix gives the final result for $\overline{\nu}\nu \to 5\gamma$:

$$\begin{aligned}\sigma(\nu\overline{\nu} \to 5\gamma) &= \frac{1487}{(2\pi)^6 2^4 3^9 5^4 7^4}\, \alpha^5 m_e^2 G_F^2 \left(\frac{1}{2} + 2s_w^2\right)^2 \left(\frac{s}{m_e^2}\right)^9 \\ &\simeq 6.87 \cdot 10^{-38} \left(\frac{s}{m_e^2}\right)^9 \text{ barn,}\end{aligned} \tag{15}$$

where $s = -(p+\overline{p})^2 = -2p\cdot\overline{p}$ is the usual Mandelstam variable, equal to $s = 4E_\nu^2$ in the centre-of-mass frame.

- $\nu\gamma \to \nu + 4\gamma$*:* A similar exercise, after crossing the external lines, yields

$$\begin{aligned}\sigma(\nu\gamma \to \nu + 4\gamma) &= \sigma(\overline{\nu}\gamma \to \overline{\nu} + 4\gamma) \\ &= \frac{13 \cdot 163 \cdot 2339}{(2\pi)^6\ 2^8 3^{10} 5^5 7^4 11} \alpha^5 m_e^2 G_F^2 \left(\frac{1}{2} + 2s_w^2\right)^2 \left(\frac{s}{m_e^2}\right)^9 \\ &\simeq 8.68 \cdot 10^{-38} \left(\frac{s}{m_e^2}\right)^9 \text{ barn.}\end{aligned} \tag{16}$$

- $\gamma\gamma \to \overline{\nu}\nu + 3\gamma$*:* Crossing the other neutrino yields,

$$\begin{aligned}\sigma(\gamma\gamma \to \overline{\nu}\nu + 3\gamma) &= \frac{797549}{(2\pi)^6\ 2^7 3^9 5^5 7^4 11}\, \alpha^5 m_e^2 G_F^2 \left(\frac{1}{2} + 2s_w^2\right)^2 \left(\frac{s}{m_e^2}\right)^9 \\ &\simeq 8.38 \cdot 10^{-38} \left(\frac{s}{m_e^2}\right)^9 \text{ barn.}\end{aligned} \tag{17}$$

## 4. Conclusion

We have constructed the effective interaction which governs the interactions of five photons and two neutrinos using the general connection between the effective action for

neutrino-photon interactions at lowest order in $G_F$ and the known Euler-Heisenberg effective interaction for photon-photon scattering. As an application we have computed the two-body cross sections whose low-energy limits are given in terms of this effective interaction. While these cross sections are likely to be too small to be of any astrophysical relevance, it is interesting that the effective interaction for such a high-order process can be obtained with such minimal effort. It is also noteworthy that a seventh-order process can compete with the two-body scattering $\overline{\nu}\nu \to 2\gamma$. Finally, our expressions were obtained by evaluating multi-body phase space integrals, for which we have presented an efficient method of computation.

## 5. Acknowledgements

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## Appendix I. Phase Space Integrals

Our goal in this appendix is to describe how to evaluate the integral, eq. (14), which we reproduce once more for convenience:

$$\mathcal{I}_m^{\alpha_1\ldots\alpha_m;\gamma_1\cdots\gamma_m}(w) = \int \frac{d^3k_1}{2k_1^0} \cdots \frac{d^3k_m}{2k_m^0}\, k_1^{\alpha_1} k_1^{\gamma_1} \cdots k_m^{\alpha_m} k_m^{\gamma_m}\ \delta^4(\sum_{i=1}^{m} k_i - w).$$

We will find, through the integral representation of the delta function, that we are able to reduce the problem to one of taking derivatives of a suitable (simple) integral. Since the general expressions are extremely lengthy, and particular results are not difficult to obtain with the help of a symbol manipulation program once the prescription is known, we will only provide a detailed recipe for evaluating these integrals.

We proceed by using the Fourier representation of the delta function to factorise this integral into products of terms having the form

$$J^{\alpha\beta}(x) = \int \frac{d^3k}{2k^0} k^\alpha k^\beta e^{ik\cdot x},$$

so that

$$\mathcal{I}_m^{\alpha_1\ldots\alpha_m;\gamma_1\cdots\gamma_m}(w) = \int \frac{d^4x}{(2\pi)^4}\, J^{\alpha_1\gamma_1}(x)\cdots J^{\alpha_m\gamma_m}(x)e^{-iw\cdot x}. \tag{18}$$

The integral defining $J^{\alpha\beta}$ is easily performed as follows:

$$J^{\alpha\beta}(x) = \frac{1}{(i)^2}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\int \frac{d^3k}{2k^0}\, e^{ik\cdot x} = \frac{4\pi}{x^6}(\eta^{\alpha\beta}x^2 - 4x^\alpha x^\beta), \tag{19}$$

where we use the integral

$$\int \frac{d^3k}{2k^0}\, e^{ik\cdot x} = \frac{2\pi}{x^2}. \tag{20}$$

We have ensured the convergence of this integral through the appropriate $\epsilon$ prescription, taking $x^2 = -(x^0)^2 + \mathbf{x}^2 + i\ sgn(x^0)\epsilon$. The $\epsilon$ term in $x^2$ forces the incoming momentum to be future-pointing, and allows the $x$ integrals to be done unambiguously. Notice that our result for $J^{\alpha\beta}$ is traceless, as is required, since the $k$s are null.

With $J^{\alpha\beta}$ in hand, expand the integrand of (18) to obtain a sum of integrals of the form

$$\int d^4x\, x^{\alpha_1}\cdots x^{\alpha_n}\frac{e^{-iw\cdot x}}{(x^2)^m},$$

where $w$ is a future-pointed, timelike four-vector. These integrals can all be evaluated by differentiating

$$\mathcal{I}_m(w) := \int d^4x\frac{e^{-iw\cdot x}}{(x^2)^m}, \tag{21}$$

with respect to $w$, so that finding this integral reduces the problem of calculating (14) to one of expanding a polynomial and taking derivatives.

Let us evaluate (21). If we explicitly factor the denominator, go to the rest frame of $w$, and consider the $t$ integral first, we need to consider

$$\begin{aligned}
\mathcal{I}_m(w) &= \int d^2\Omega \int_0^\infty dr r^2 \int_{-\infty}^\infty dt\frac{e^{+i\omega t}}{[\,-(t-i\epsilon)+r]^m[\,(t-i\epsilon)+r]^m} \\
&= \frac{1}{2}\int d^2\Omega \int_{-\infty}^\infty dr\ r^2 \int_{-\infty}^\infty dt\ \frac{e^{+i\omega t}}{(-)^m[\,t-(r+i\epsilon)]^m[\,t+(r-i\epsilon)]^m} \\
&= 2\pi(-)^m \int_{-\infty}^\infty dr\ r^2 \int_{-\infty}^\infty dt\ \frac{e^{+i\omega t}}{[\,t-(r+i\epsilon)]^m[\,t-(-r+i\epsilon)]^m}.
\end{aligned}$$

The $t$ integral in this expression is a contour integral, which is nonzero only for $\omega > 0$, where $\omega = w^0$. For $\omega > 0$ we close the contour upwards in a semicircle, break it into two, one around each pole, and use the Cauchy integral formula to get

$$\begin{aligned}\mathcal{I}_m(w) &= \int_{-\infty}^{\infty} dt\, \frac{e^{+i\omega t}}{[\,t-(r+i\epsilon)]^m[\,t-(-r+i\epsilon)]^m} \\ &= \frac{2\pi i}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}\left(\frac{e^{iz\omega}}{[z-(i\epsilon-r)]^m}\right)_{z=r+i\epsilon} + \frac{d^{m-1}}{dz^{m-1}}\left(\frac{e^{iz\omega}}{[z-(r+i\epsilon)]^m}\right)_{z=-r+i\epsilon}\right\}\theta(\omega).\end{aligned}$$

Using the Leibniz rule, $d^m/dz^m(f(z)\,g(z)) = \sum_{s=0}^{n}\binom{m}{s} f^{(n-s)}(z)g^{(s)}(z)$, yields,

$$\begin{aligned}\mathcal{I}_m(w) = &\frac{2\pi i}{[(m-1)!]^2\,2^m}\sum_{s=0}^{m-1}\binom{m-1}{s}(-)^s\,(m+s-1)! \\ &\times\frac{(i\omega)^{m-1-s}}{2^s}\left[\frac{e^{ir\omega}}{r^{m+s}} + (-)^{m+s}\,\frac{e^{-ir\omega}}{r^{m+s}}\right]\theta(\omega).\end{aligned} \tag{22}$$

Now using this in eq. (22) yields, after letting $r \leftrightarrow -r$ in the second term of (22),

$$\begin{aligned}\mathcal{I}_m(w) = &\frac{8\pi^2 i}{[(m-1)!]^2\,2^m}\sum_{s=0}^{m-1}\binom{m-1}{s}(-)^{m-s}\,(m+s-1)! \\ &\times\theta(\omega)\frac{(i\omega)^{m-1-s}}{2^s}P\int_{-\infty}^{\infty} dr\,\frac{e^{ir\omega}}{r^{m+s-2}}.\end{aligned} \tag{23}$$

Finally, since

$$P\int_{-\infty}^{\infty} dr\,\frac{e^{ir\omega}}{r^n} = \frac{i\pi(i\omega)^{n-1}}{(n-1)!},$$

substitution into (23) yields

$$\mathcal{I}_m(w) = (-)^{m+1}\theta(\omega)\,\frac{8\pi^3(i\omega)^{2m-4}}{[(m-1)!]^2\,2^m}\sum_{s=0}^{m-1}\binom{m-1}{s}\frac{(-)^s}{2^s}\,(m+s-1)(m+s-2).$$

The sum is easily recognised as the second derivative of $2^{m-3}\,[x(1-x)]^{m-1}$, evaluated at $x=\frac{1}{2}$, and equal to $\frac{-2(m-1)}{2^{m-1}}$, so we obtain the general expression,

$$\mathcal{I}_m(w) = \theta(\omega)\,\frac{32\pi^3(m-1)}{[(m-1)!]^2\,4^m}(-w^2)^{m-2},\ \text{for } m\geq 3.$$

So, to summarise, by replacing the delta function by an integral, we are able to do each of the $k$ integrals separately, obtaining (18). Collecting the terms in the expansion and using (21) and its derivatives to proceed with the $x$ integrals yields the final answer to (14).

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